

Hard-core thinnings of germ–grain models with power-law grain sizes

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Abstract

Random sets with long-range dependence can be generated using a Boolean model with power-law grain sizes. We study thinnings of such Boolean models which have the hard-core property that no grains overlap in the resulting germ–grain model. A fundamental question is whether long-range dependence is preserved under such thinnings. To answer this question we study four natural thinnings of a Poisson germ–grain model where the grains are spheres with a regularly varying size distribution. We show that a thinning which favors large grains preserves the slow correlation decay of the original model, whereas a thinning which favors small grains does not. Our most interesting finding concerns the case where only disjoint grains are retained, which corresponds to the well-known Matérn type I thinning. In the resulting germ–grain model, typical grains have exponentially small sizes, but rather surprisingly, the long-range dependence property is still present. As a byproduct, we obtain new mechanisms for generating homogeneous and isotropic random point configurations having a power-law correlation decay.

1 Introduction

Consider a random closed set which can be expressed as a union of compact sets in the d -dimensional Euclidean space \mathbb{R}^d . The compact building blocks of the random set are called grains, the collection of grains *germ–grain model*, and the union of grains *grain cover*. A germ–grain model is called *hard-core* if the grains are disjoint with probability one. Hard-core germ–grain models (a.k.a. random packing models) provide an important class of mathematical tools for the natural sciences, allowing to model and analyze the statistical features of disordered porous materials [OM00, SSW02]. Besides natural sciences, these models have found applications in engineering when analyzing the performance of medium access protocols in wireless data networks (e.g. [BB09, Hae11, NB12]).

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A key statistical feature of a random set is its covariance function, which describes how much more or less likely it is to find matter at a given distance from a location containing matter, compared to finding matter in an arbitrary location. While most germ–grain models studied in the literature have a rapidly decaying covariance function, certain experimental studies in astronomy [JMST05] and materials science [SSW02] display real-world data where the statistically estimated covariance function appears to decay exceptionally slowly, following a power law $r^{-\beta}$ with some exponent $\beta > 0$ for large distances r . When $\beta < d$, such models are *long-range dependent* in the sense that

$$\limsup_{r \rightarrow \infty} \frac{\text{Var}(|X \cap B_r|)}{r^d} = \infty, \quad (1.1)$$

where $|X \cap B_r|$ denotes the volume of the region covered by the random set X within the closed ball B_r with radius r centered at the origin [DVJ08, Sec 12.7]. Long-range dependence causes anomalous behavior to several statistical features of the model, as is well understood in time series analysis [Sam06]. Note that for a homogeneous random set in dimension $d = 1$, property (1.1) is equivalent to the usual notion of long-range dependence,

$$\limsup_{n \rightarrow \infty} \frac{\text{Var}(\sum_{k=1}^n X_k)}{n} = \infty,$$

of the time series $X_k = |X \cap (k-1, k]|$.

Our goal in this article is to construct parsimonious germ–grain models having the hard-core and long-range dependence property. In the presence of long-range dependence, the requirement of parsimony, i.e. having a small number of model parameters, is especially important because long-range dependence tends to reduce the robustness of the statistical estimators of model parameters [CSN09]. Long-range dependent germ–grain models are easy to generate using a Boolean model—a germ–grain model with random power-law distributed sizes and independently and uniformly scattered centers—but the resulting model is not hard-core by construction. To make it hard-core, we shall follow Matérn’s approach [Mat60] of thinning out a selected collection of overlapping grains from the proposed Boolean model so that the resulting collection of grains is disjoint. Whether this approach is feasible for obtaining hard-core models with long-range dependence depends on the following question:

Is the power-law covariance decay of the proposed Boolean model preserved after making it disjoint by thinning?

To answer the above question, we shall analyze in detail the following natural thinning mechanisms:

- *Large retained.* Let the thinned model consist of those grains in the original Boolean model which are not overlapped by any larger grain in the original model.

- *Random retained.* Assign independent random weights to the grains. Let the thinned model consist of those grains in the original model which are not overlapped by any heavier grain in the original model. (This thinning corresponds to Matérn type II.)
- *Small retained.* Let the thinned model consist of those grains in the original model which are not overlapped by any smaller grain in the original model.
- *Isolated retained.* Let the thinned model be the set of grains in the original model which do not overlap with any other grain in the original model. (This thinning corresponds to Matérn type I.)

We remark that—unlike the Matérn type III hard-core model [NB12]—the above thinnings are local in that the decision whether a proposed grain shall be retained or not is made solely by looking at the grains which intersect it.

For simplicity, we shall restrict to spherical models where the grains are closed balls. Figure 1 illustrates the above four thinnings applied to a simulated sample of a Boolean model in \mathbb{R}^2 where the grain centers have mean density $\lambda = 0.05$ and the grain radii have a Pareto distribution $F(r) = 1 - r^{-\alpha}$, $r \geq 1$, with tail exponent $\alpha = 2.5$.

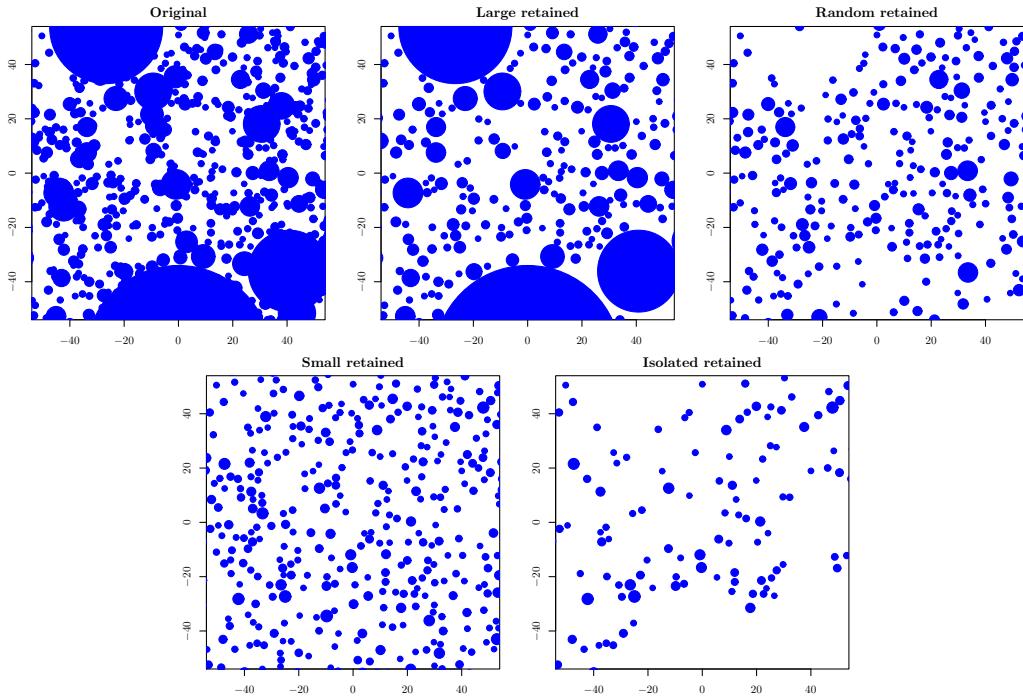


Figure 1: Original model and its four hard-core thinnings, where the original model is generated by disks having a Pareto distribution with tail exponent $\alpha = 2.5$.

The above thinnings will be analyzed collectively by viewing them as instances

of a general weight-based thinning mechanism, following Månsson and Rudemo [MR02]. Standard formulas of Palm calculus allow to write down closed-form analytical formulas for the radius distribution of a typical grain, the covariance function of the grain cover, and the two-point correlation function of the grain centers for general hard-core germ–grain configurations generated by weight-based thinnings. Using the theory of regular variation, we analyze the long-range behavior of these quantities under the assumption that the grain radii in the proposed Boolean model follow a power-law distribution with tail exponent $\alpha > d$.

The main results of this article (Theorems 5.1, 6.1, 7.1, 8.1) are summarized in Table 1 below. From the table, we can draw the following conclusions:

Model	Radius distribution of a typical grain	Covariance function of grain cover	Correlation function of grain centers
Original	power law (α)	power law ($\alpha - d$)	zero
Large retained	power law (α)	power law ($\alpha - d$)	power law ($\alpha - d$)
Random retained	power law ($\alpha + d$)	power law ($\alpha - d$)	power law ($\alpha - d$)
Small retained	exponential	exponential	exponential
Isolated retained	exponential	power law ($\alpha - d$)	power law ($\alpha - d$)

Table 1: Long-range decay of key statistical characteristics of the original Boolean model and the hard-core germ–grain models obtained by thinning.

- The power-law covariance decay and long-range dependence (when $\alpha < 2d$) of the grain cover are preserved under all thinnings except *small retained*.
- Whereas the random point configuration (a.k.a. point process) of grain centers in the proposed Boolean model is completely uncorrelated, the corresponding point configurations in all thinned models except *small retained* have a power-law two-point correlation function.
- The heavy tail of the grain radius distribution is destroyed by *small retained* and *isolated retained* thinnings. The other two thinnings preserve the power-law structure of the tail distribution: under *large retained* with the same exponent, under *random retained* with a larger exponent corresponding to a lighter tail.

Table 1 also reveals a striking feature of the *isolated retained* thinning mechanism: The resulting grain cover and the resulting point configuration of grain centers both exhibit long-range dependence although the grain size distribution is light-tailed. This seemingly paradoxical phenomenon can be explained by inspecting the empty space: Any region of space not covered by the thinned germ–grain model is likely to have been contained in a big grain of the proposed model that was removed in the thinning, and therefore, a large neighborhood of this empty region is likely to be empty, too.

This article may be seen as a continuation of the works of Månsson and Rudemo [MR02] and Andersson, Häggström and Månsson [AHM06], who analyzed first-

order statistical properties of hard-core germ–grain models obtained by weight-based thinnings. In [MR02, Cor 3.1] it was also shown that *large retained* thinning preserves the tail behavior of the typical grain radius whenever the proposed grain radius distribution is continuous. A slightly more general thinning framework was recently introduced by Nguyen and Baccelli [NB12], who derived differential equations characterizing the generating functional of the random point configuration formed by the thinned grain centers. Earlier work on the covariance analysis of random sets includes Böhm and Schmidt [BS03], who derived a short-range approximation for the covariance function of a general homogeneous random set. Snethlage, Martínez, Stoyan, Saar [MVDE02] (see also references therein) provide a nice summary of random point configuration models where the two-point correlation function has a power-law behavior on short distances. Earlier works on long-range dependent random sets appear mostly restricted to random point configuration in dimension $d = 1$. Among these, Daley and Vesilo [DV97] established the following elegant preservation property for many queueing systems: the point configuration of the departure times is long-range dependent if and only if the same is true for the arrival times. Daley [Da99] showed that a renewal point process is long-range dependent if the interpoint distances have an infinite second moment, and Kulik and Szekli [KS01] extended this observation to one-dimensional point configurations with positively associated interpoint distances. Vamvakos and Anantharam [VA98] showed that the long-range dependence of a point process is preserved by a leaky bucket flow control mechanism for data traffic. A study focused on the long-range dependence of multidimensional random sets is the recent work of Demichel, Estrade, Kratz, and Samorodnitsky [DEKS11], who studied whether random sets having power-law decaying chord length distributions, closely related to the covariance function of the random set, can be generated as a level set of a Gaussian random field—they found that in wide generality (merely assuming that the underlying Gaussian field is mixing), this is not possible.

Let us summarize the notational conventions used in this paper. The symbol P stands for the probability measure on some abstract probability space which governs all randomness in the models, and $E, \text{Var}, \text{Cov}$ denote the expectation, variance, and covariance with respect to P , respectively. The symbol $B_r(x)$ denotes the closed unit ball with center x and radius r in the d -dimensional Euclidean space \mathbb{R}^d . We use B_r as shorthand for $B_r(o)$, where o is the origin of \mathbb{R}^d . For a Borel set B in \mathbb{R}^d , we denote by $|B|$ its Lebesgue measure, and by $\mathbf{1}_B(x)$ or $\mathbf{1}(x \in B)$ its indicator function. The symbols dx, dy , etc. refer to the Lebesgue measure in \mathbb{R}^d . The symbol \mathbb{R}_+ denotes the positive real numbers including zero. The symbol $F(dr)$ refers to integration with respect to a probability measure F on \mathbb{R}_+ , whereas $F(r) = F[0, r]$ and $\bar{F}(r) = 1 - F(r)$ stand for the corresponding cumulative distribution function and the complementary cumulative distribution function, respectively. The minimum and maximum of real numbers a and b are denoted by $a \wedge b$ and $a \vee b$, respectively. When convenient, we denote $\int_a^\infty = \int_{(a, \infty)}$, $\int_0^b = \int_{[0, b]}$, and $\int_a^b = \int_{(a, b]}$ for $0 < a < b < \infty$. For functions f and g defined on the

positive real line, we denote $f \sim g$ if $f(t)/g(t) \rightarrow 1$ as $t \rightarrow \infty$.

The rest of the paper is organized as follows. Section 2 summarizes preliminaries on random Boolean models needed later in the text. Section 3 introduces a weight-based thinning mechanism which produces hard-core germ–grain models from Boolean models and list formulas for the second-order statistics of the models so obtained. Section 4 contains a long-range analysis of the second-order statistics of the previous section. The main results of Table 1 are proved case-by-case in Section 5 (isolated retained), Section 6 (random retained), Section 7 (large retained), and Section 8 (small retained). Section 9 concludes the paper.

2 Boolean models with power-law grain radii

A spherical Boolean model is a random collection of closed spheres, where the sphere centers are independently and uniformly scattered in \mathbb{R}^d and the sphere radii are independent and identically distributed random variables in \mathbb{R}_+ . Mathematically, a spherical Boolean model can be defined as a Poisson random measure Φ on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda dx F(dr)$, where λ is a positive constant and F is a probability measure on \mathbb{R}_+ such that $\int r^d F(dr) < \infty$. We identify each pair $(x, r) \in \Phi$ with the closed ball $B_r(x)$ with center x and radius r and—conforming to the terminology of more general germ–grain models—such pairs will be called *grains*. The random closed set

$$X = \bigcup_{(x, r) \in \Phi} B_r(x)$$

is called the *grain cover* of Φ , and we denote by

$$\Phi_g = \{x \in \mathbb{R}^d : (x, r) \in \Phi \text{ for some } r\}$$

the random point configuration in \mathbb{R}^d formed by the grain centers of Φ . Note that Φ_g is a homogeneous Poisson random measure on \mathbb{R}^d with intensity measure λdx . The parameter λ thus equals the mean density of grain centers, and the probability measure F is the common distribution of grain radii. For general definitions and details about random sets and random measures, see for example [DVJ08, Mol05, SW08, SKM95].

The covariances of the random set X are denoted by $k(x, y) = \text{Cov}(\mathbf{1}_X(x), \mathbf{1}_X(y))$, where $\mathbf{1}_X$ is the indicator function of X . Because the distribution of X is shift-invariant by construction, the covariances are given by $k(x, y) = k(x - y)$, where the covariance function $k(z) = k(o, z)$ is given by the well-known formula (e.g. [SKM95, Sec. 3.1])

$$k(z) = (1 - p)^2 \left(e^{\lambda \int |B_r(o) \cap B_r(z)| F(dr)} - 1 \right), \quad (2.1)$$

and where p is the volume fraction of X given by

$$p = 1 - e^{-\lambda \int |B_r| F(dr)}.$$

Formula (2.1) indeed shows that $k(z)$ depends on z only through $|z|$, which is evident because X is isotropic by construction. Using this formula we may also deduce that

$$k(z) \sim \lambda(1-p)^2 \int |B_r(o) \cap B_r(z)| F(dr) \quad \text{as } |z| \rightarrow \infty,$$

where we denote $f(z) \sim g(z)$ if $f(z)/g(z) \rightarrow 1$ as $|z| \rightarrow \infty$. When the grain radius distribution F follows a power law with tail exponent $\alpha > d$, so that $F(r) = 1 - \ell(r)r^{-\alpha}$ for some slowly varying function ℓ (see Appendix A for details), it follows by using Lemma B.1 in Appendix B that

$$k(z) \sim \lambda(1-p)^2 c_{\alpha,d} \ell(|z|) |z|^{-(\alpha-d)} \quad \text{as } |z| \rightarrow \infty.$$

Thus, when the radius distribution follows a power law with tail exponent $\alpha > d$, then the covariance function $k(z)$ follows a power law with tail exponent $\alpha - d$. Especially, the Boolean grain cover X is long-range dependent in the sense of (1.1) for $\alpha \in (d, 2d)$.

3 Weight-based thinning

In this section we shall study a weight-based thinning mechanism which maps a Boolean model into a hard-core germ–grain model consisting of nonoverlapping grains [MR02, NB12]. This thinning mechanism is defined by assigning random weights to the grains of the Boolean model, and retaining those grains which are not overlapped by any other grain in the Boolean model with a higher or equal weight.

3.1 Thinning mechanism

A weighted spherical Boolean model is defined as a Poisson random measure Φ on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure

$$\Lambda(dx, dr, dw) = \lambda dx F(dr) G_r(dw),$$

where $\lambda > 0$, F is a probability measure on \mathbb{R}_+ such that $\int r^d F(dr) < \infty$, and G is a probability kernel on \mathbb{R}_+ (a family of probability measures G_r on \mathbb{R}_+ indexed by r such that $r \mapsto G_r(A)$ is measurable for measurable $A \subset \mathbb{R}_+$). A triplet $(x, r, w) \in \Phi$ is identified as a grain with center x , radius r , and weight w . As in Section 2, the constant λ is the mean density of grain centers and the probability measure F is the distribution of grain radii. The probability measure G_r is the weight distribution of a grain with radius r .

We say that two distinct grains are *neighbors* if they intersect each other, and we denote the set of neighbors of a reference grain (x, r, w) by

$$N_{x,r,w} = \left\{ (x', r', w') \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, r, w)\} : B_{r'}(x') \cap B_r(x) \neq \emptyset \right\}. \quad (3.1)$$

The thinning of a weighted spherical Boolean model Φ is now defined by $\Phi^{\text{th}} = T(\Phi)$, where

$$T(\Phi) = \{(x, r, w) \in \Phi : w > w' \text{ for all } (x', r', w') \in \Phi \cap N_{x, r, w}\}. \quad (3.2)$$

To rephrase the definition, we say that a grain (x', r', w') *obstructs* grain (x, r, w) if (x', r', w') is a neighbor of (x, r, w) and $w' \geq w$. Then by definition, the thinned germ–grain configuration Φ^{th} consists of grains in Φ which are not obstructed by any other grain in Φ . Note that two overlapping grains with equal weights obstruct each other, and will be both removed.

The following choices of G_r yield the four thinnings which shall be analyzed in detail in Sections 5–8.

- *Large retained.* The weight of each grain is set equal to its radius, so that $G_r(dw) = \delta_r(dw)$.
- *Random retained.* The grains are assigned independent uniformly distributed random weights, so that $G_r(dw) = 1_{(0,1)}(w)dw$.
- *Small retained.* The weight of each grain is set equal to the inverse of its radius, so that $G_r(dw) = \delta_{1/r}(dw)$.
- *Isolated retained.* All grains are assigned weight one, so that $G_r(dw) = \delta_1(dw)$.

3.2 Retention probability

The *retention probability* of a reference grain (x, r, w) is defined as the probability that (x, r, w) belongs to the germ–grain configuration obtained by thinning the union $\Phi \cup \{(x, r, w)\}$. Because this probability does not depend on x (see Proposition 3.1), we shall denote it by

$$h(r, w) = \mathbb{P}((x, r, w) \in T(\Phi \cup \{(x, r, w)\})).$$

The quantity $h(r, w)$ may be regarded as the probability that a typical grain with radius r and weight w in the proposed Boolean model is retained (see e.g. [SW08, SKM95]). Analogously, the weight-averaged retention probability

$$h(r) = \int_{\mathbb{R}_+} h(r, w) G_r(dw) \quad (3.3)$$

may be regarded as the probability that a typical grain of radius r in the proposed Boolean model is retained. The following result [MR02, Thm. 2.2] gives a formula for the retention probability. For the reader’s convenience we will include the proof here.

Proposition 3.1. *The retention probability of an arbitrary reference grain $(x, r, w) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ does not depend on x , and is given by*

$$h(r, w) = \exp \left\{ -\lambda \int_{\mathbb{R}_+} |B_{r+s}(o)| G_s[w, \infty) F(ds) \right\}. \quad (3.4)$$

Proof. Fix a reference grain (x, r, w) and denote $\Phi' = \Phi \cup \{(x, r, w)\}$. By definition, the reference grain belongs to the thinned configuration $T(\Phi')$ if and only if $w > w'$ for all $(x', r', w') \in N_{x,r,w} \cap \Phi'$, where $N_{x,r,w}$ is the neighbor set of (x, r, w) defined by (3.1). Observe that $N_{x,r,w} \cap \Phi' = N_{x,r,w} \cap \Phi$, because no grain is its own neighbor by definition. As a consequence, the retention probability can be expressed using the the intensity measure of the Poisson point configuration Φ according to

$$\mathbb{P}((x, r, w) \in T(\Phi')) = \mathbb{P}(\Phi(A_{x,r,w}) = 0) = e^{-\Lambda(A_{x,r,w})},$$

where

$$A_{x,r,w} = \{(x', r', w') \in N_{x,r,w} : w' \geq w\}$$

is the set of grains obstructing (x, r, w) . The claim now follows because

$$\begin{aligned} \Lambda(A_{x,r,w}) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbf{1}(|x - x'| \leq r + r') \mathbf{1}(w' \geq w) G_{r'}(dw') F(dr') \lambda dx' \\ &= \lambda \int_{\mathbb{R}_+} |B_{r+r'}(o)| G_{r'}[w, \infty) F(dr'). \end{aligned}$$

□

3.3 First-order statistics of the thinned model

Let us summarize some key formulas about the first-order statistics of the thinned germ-grain model Φ^{th} which were obtained in [AHM06, MR02]. The mean density of grain centers in the thinned model is given by

$$\lambda_{\text{th}} = \lambda \int_{\mathbb{R}_+} h(r) F(dr), \quad (3.5)$$

where $h(r)$ is the weight-averaged retention probability defined in (3.3), and the radius distribution of a typical grain in the thinned model equals

$$F_{\text{th}}(r) = 1 - \frac{\lambda}{\lambda_{\text{th}}} \int_r^\infty h(s) F(ds). \quad (3.6)$$

Moreover, the volume fraction of the thinned grain cover

$$X^{\text{th}} = \bigcup_{(x,r,w) \in \Phi^{\text{th}}} B_r(x)$$

is given by

$$p_{\text{th}} = \lambda \int_{\mathbb{R}_+} |B_r| h(r) F(dr). \quad (3.7)$$

Note that the quantity $\int h(r) F(dr)$ in (3.5) may be regarded as the probability that a randomly chosen grain in the proposed Boolean model is retained by the thinning mechanism.

3.4 Pair retention probability

The *pair retention probability* of a given pair of reference grains (x_1, r_1, w_1) and (x_2, r_2, w_2) is defined as the probability that both reference grains belong to the germ-grain configuration obtained by thinning the union $\Phi' = \Phi \cup \{(x_1, r_1, w_1), (x_2, r_2, w_2)\}$. Because this probability depends on x_1 and x_2 only through their distance (see Proposition 3.2), we shall denote it by

$$h_2(u, r_1, w_1, r_2, w_2) = P(\{(x_1, r_1, w_1), (x_2, r_2, w_2)\} \in T(\Phi')), \quad (3.8)$$

where $u = |x_1 - x_2|$. The weight-averaged pair retention probability is defined by

$$h_2(u, r_1, r_2) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} h_2(u, r_1, w_1, r_2, w_2) G_{r_1}(dw_1) G_{r_2}(dw_2). \quad (3.9)$$

Proposition 3.2. *The pair retention probability of two reference grains (x_1, r_1, w_1) and (x_2, r_2, w_2) depends on x_1 and x_2 only through the distance $u = |x_1 - x_2|$. For $u \leq r_1 + r_2$ this probability equals zero, and for $u > r_1 + r_2$,*

$$h_2(u, r_1, w_1, r_2, w_2) = h(r_1, w_1) h(r_2, w_2) e^{\tau(u, r_1, w_1, r_2, w_2)}$$

where $h(r_1, w_1)$ and $h(r_2, w_2)$ are the retention probabilities defined by (3.4), and

$$\tau(u, r_1, w_1, r_2, w_2) = \lambda \int_{\mathbb{R}_+} |B_{r+r_1}(x_1) \cap B_{r+r_2}(x_2)| G_r[w_1 \vee w_2, \infty) F(dr)$$

is the mean number of grains in Φ which simultaneously obstruct both reference grains.

Proof. Fix two reference grains (x_1, r_1, w_1) and (x_2, r_2, w_2) and assume that they do not overlap, so that $|x_1 - x_2| > r_1 + r_2$. Denote $\Phi' = \Phi \cup \{(x_1, r_1, w_1), (x_2, r_2, w_2)\}$. Recall that grain (x_1, r_1, w_1) belongs to $T(\Phi')$ if and only if $w_1 > w$ for all $(x, r, w) \in N(x_1, r_1, w_1) \cap \Phi'$. Because no grain is its own neighbor by definition, and because the two reference grains are not neighbors, we see that $N(x_1, r_1, w_1) \cap \Phi' = N(x_1, r_1, w_1) \cap \Phi$. By symmetry, a similar conclusion also holds for the other reference grain.

We conclude that for $i = 1, 2$, grain (x_i, r_i, w_i) is retained if and only if $\Phi(A_i) = \emptyset$, where

$$A_i = \{(x, r, w) \in N(x_i, r_i, w_i) : w \geq w_i\}$$

is the set of grains obstructing (x_i, r_i, w_i) . Now the pair retention probability can be written as

$$h_2 = P(\Phi(A_1 \cup A_2) = 0). \quad (3.10)$$

The number of grains in $\Phi \cap (A_1 \cup A_2)$ is Poisson distributed with mean

$$\Lambda(A_1 \cup A_2) = \Lambda(A_1) + \Lambda(A_2) - \Lambda(A_1 \cap A_2).$$

Because $e^{-\Lambda(A_i)}$ equals the retention probability $h(r_i, w_i)$ of grain (x_i, r_i, w_i) (see Proposition 3.1), we see that

$$h_2 = h(r_1, w_1)h(r_2, w_2)e^{\Lambda(A_1 \cap A_2)}.$$

The claim now follows after noting that

$$\Lambda(A_1 \cap A_2) = \lambda \int_{\mathbb{R}_+} |B_{r_1+r}(x_1) \cap B_{r_2+r}(x_2)| G_r[w_1 \vee w_2, \infty) F(dr).$$

□

A key quantity for analyzing the covariance function of the thinned grain cover in Section 4 is the following function, which we shall call the *retention covariance function*. It is defined by

$$q(u, r_1, r_2) = h_2(u, r_1, r_2) - h(r_1)h(r_2), \quad (3.11)$$

where $h(r)$ denotes the weight-averaged retention probability defined in (3.3), and $h_2(u, r_1, r_2)$ is the weight-averaged pair retention probability defined in (3.9).

Lemma 3.3. *The retention covariance function satisfies*

$$|q(u, r_1, r_2)| \leq h(r_1) \wedge h(r_2)$$

for all $u, r_1, r_2 \geq 0$.

Proof. Fix a pair of reference grains (x_1, r_1, w_1) and (x_2, r_2, w_2) having their centers at a distance $u = |x_1 - x_2|$ apart. Define a weight-dependent version of q by

$$q(u, r_1, w_1, r_2, w_2) = h_2(u, r_1, w_1, r_2, w_2) - h(r_1, w_1)h(r_2, w_2).$$

We will first show that

$$|q(u, r_1, w_1, r_2, w_2)| \leq h(r_1, w_1), \quad (3.12)$$

by separately considering the following two cases:

- (i) If $u \leq r_1 + r_2$, then $h_2(u, r_1, w_1, r_2, w_2)$ is zero because the reference grains overlap, and (3.12) follows immediately.

(ii) If $u > r_1 + r_2$, then by borrowing the notation from the proof of Proposition 3.2, we have by (3.10) that

$$0 \leq h_2(u, r_1, w_1, r_2, w_2) = \mathbb{P}(\Phi(A_1 \cup A_2) = 0) \leq \mathbb{P}(\Phi(A_1) = 0) = h(r_1, w_1).$$

As a consequence,

$$-h(r_1, w_1)h(r_2, w_2) \leq q(u, r_1, w_1, r_2, w_2) \leq h(r_1, w_1)(1 - h(r_2, w_2)),$$

from which (3.12) again follows.

After integrating both sides of (3.12) over the weights, we see that $|q(u, r_1, r_2)| \leq h(r_1)$. By symmetry, the same inequality holds with r_1 replaced by r_2 , which proves the claim. \square

3.5 Covariance function of the thinned grain cover

Let us now consider the covariance function

$$k_{\text{th}}(z) = \mathbb{P}(o \in X^{\text{th}}, z \in X^{\text{th}}) - \mathbb{P}(o \in X^{\text{th}}) \mathbb{P}(z \in X^{\text{th}})$$

of the thinned grain cover X^{th} .

Proposition 3.4. *The covariance function of the thinned grain cover is given by*

$$\begin{aligned} k_{\text{th}}(z) &= \lambda \int_{\mathbb{R}_+} |B_r(o) \cap B_r(z)| h(r) F(dr) \\ &+ \lambda^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} |B_{r_1}(o) \cap B_{r_2}(z)| q(|x - z|, r_1, r_2) dx F(dr_1) F(dr_2), \end{aligned} \quad (3.13)$$

where h is the weight-averaged retention probability defined by (3.3) and q is the retention covariance function defined by (3.11).

Proof. Let us express the covariance function as

$$k_{\text{th}}(z) = S_1(z) + S_2(z) - p_{\text{th}}^2,$$

where $S_1(z)$ is the probability that a single grain in Φ^{th} simultaneously covers o and z , $S_2(z)$ is the probability that o and z are covered by distinct grains in Φ^{th} , and the volume fraction p_{th} can be viewed as the probability that an arbitrary reference point in \mathbb{R}^d is covered by some grain in Φ^{th} .

To write down an analytical expression for $S_1(z)$, recall first that by the hardcore property, the indicator function of X^{th} can be written as

$$\mathbf{1}_{X^{\text{th}}}(y) = \sum_{(x, r, w) \in \Phi} f_y(\Phi; x, r, w),$$

where $f_y(\Phi; x, r, w) = \mathbf{1}(y \in B_r(x))\mathbf{1}_{T(\Phi)}(x, r, w)$ is the indicator for the event that a grain (x, r, w) covers y and is contained in Φ^{th} . Then

$$S_1(z) = \mathbb{E} \sum_{(x, r, w) \in \Phi} f_o(\Phi; x, r, w) f_z(\Phi; x, r, w),$$

Using Mecke's formula [SW08, Thm. 3.2.5] it's easy to see that

$$S_1(z) = \lambda \int_{\mathbb{R}_+} |B_r(o) \cap B_r(z)| h(r) F(dr),$$

where $h(r) = \int h(r, w) G_r(dw)$.

The probability that o and z are covered by distinct grains in Φ^{th} can analogously be written as

$$S_2(z) = \mathbb{E} \sum_{\substack{(x_1, r_1, w_1) \in \Phi \\ (x_2, r_2, w_2) \in \Phi \\ (x_1, r_1, w_1) \neq (x_2, r_2, w_2)}} f_o(\Phi; x_1, r_1, w_1) f_z(\Phi; x_2, r_2, w_2).$$

Using the Slivnyak–Mecke formula [SW08, Cor. 3.2.3], it's not hard to verify that

$$S_2(z) = \lambda^2 \iiint |B_{r_1}(o) \cap B_{r_2}(x)| h_2(|x - z|, r_1, r_2) dx F(dr_1) F(dr_2),$$

where h_2 is the pair retention probability defined by (3.8). The validity of the claim now follows after representing p_{th} using (3.7) and the identity $|B_{r_1}(o)| |B_{r_2}(o)| = \int |B_{r_1}(o) \cap B_{r_2}(x)| dx$ to note that

$$p_{\text{th}}^2 = \lambda^2 \iiint |B_{r_1}(o) \cap B_{r_2}(x)| h(r_1) h(r_2) dx F(dr_1) F(dr_2).$$

□

3.6 Two-point correlation function of thinned grain centers

The *two-point correlation function* $\xi_{\text{th}}(z)$ of the random point configuration $\Phi_g^{\text{th}} = \{x : (x, r, w) \in \Phi^{\text{th}}\}$ of the thinned grain centers is defined as a function which satisfies

$$\text{Cov}(\Phi_g^{\text{th}}(A), \Phi_g^{\text{th}}(B)) = \lambda_{\text{th}}^2 \int_A \int_B \xi_{\text{th}}(x - y) dx dy$$

for all disjoint and bounded measurable sets $A, B \subset \mathbb{R}^d$, assuming such function exists. This function, which in our case only depends on $|z|$, describes how much more ($\xi_{\text{th}}(z) > 0$) or less ($\xi_{\text{th}}(z) < 0$) likely it is to observe a point at a distance $|z|$ from a typical point, compared to observing a point in an arbitrary location. The two-point correlation function is related to the pair-correlation function g_{th} commonly used in statistics, via the formula $\xi_{\text{th}}(z) = g_{\text{th}}(z) - 1$ (e.g. [SKM95]).

Proposition 3.5. *The two-point correlation function of the thinned grain centers is given by*

$$\xi_{\text{th}}(z) = \frac{\lambda^2}{\lambda_{\text{th}}^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} q(|z|, r_1, r_2) F(dr_1) F(dr_2), \quad (3.14)$$

where λ_{th} is the thinned germ density defined in (3.5), and q is the retention covariance function defined in (3.11).

Proof. By using the Slivnyak–Mecke formula [SW08, Cor. 3.2.3] one can check that

$$\mathbb{E} \Phi_g^{\text{th}}(A) \Phi_g^{\text{th}}(B) = \lambda^2 \int_A \int_B \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} h_2(|x - y|, r_1, r_2) F(dr_1) F(dr_2) dx dy.$$

for all bounded and disjoint $A, B \subset \mathbb{R}^d$, where h_2 is the weight-averaged pair retention probability defined in (3.9). On the other hand, Mecke’s formula [SW08, Thm. 3.2.5] implies that

$$\mathbb{E} \Phi_g^{\text{th}}(A) = \lambda \int_A \int_{\mathbb{R}_+} h(r) F(dr) dx,$$

where $h(r)$ is the weight-averaged retention probability defined in (3.3). The claim follows by combining the above two formulas and recalling the definition of the retention covariance function (3.11). \square

4 Long-range behavior of second-order statistics

In this section we assume that the grain radius distribution F of the proposed Boolean model follows a power law with tail exponent $\alpha > d$, by which we mean that the complementary cumulative distribution function $\bar{F}(r) = 1 - F(r)$ is regularly varying at infinity with exponent $-\alpha$. In this case we can write

$$\bar{F}(r) = \ell(r)r^{-\alpha},$$

where the function ℓ is slowly varying at infinity (see Appendix A for details).

4.1 Asymptotic covariance

The following result describes the covariance function of the thinned grain cover for thinnings where large grains have small retention probability.

Proposition 4.1. *Assume that the radius distribution F follows a power law with tail exponent $\alpha > d$. Assume that the weight-averaged retention probability $h(r)$ decays to zero as $r \rightarrow \infty$, and that for any $r_1, r_2 \geq 0$, the retention covariance function defined in (3.11) decays according to*

$$q(|z|, r_1, r_2) \sim q_\infty(r_1, r_2) \bar{F}(|z|) |z|^d \quad \text{as } |z| \rightarrow \infty. \quad (4.1)$$

Then the covariance function of the thinned grain cover decays according to

$$k_{\text{th}}(z) \sim c\bar{F}(|z|)|z|^d \quad \text{as } |z| \rightarrow \infty,$$

where

$$c = \lambda^2 |B_1|^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} r_1^d r_2^d q_\infty(r_1, r_2) F(dr_1) F(dr_2).$$

To prove Proposition 4.1 we need detailed results about the retention probabilities. The following lemma allows us to use dominated convergence on a part of the domain.

Lemma 4.2. *Assume that the radius distribution F follows a power law with tail exponent $\alpha > d$. Then there exist constants $c > 0$ and $m > 0$ such that*

$$0 \leq q(|x - z|, r_1, r_2) \leq c|z|^d \bar{F}(|z|)$$

for all $x, z \in \mathbb{R}^d$ and all $r_1, r_2 \geq 0$ such that $|x| < r_1 + r_2$, $|x - z| \geq 2(r_1 + r_2)$, and $|z| > m$.

Proof. Let c_1 and u_1 be the constants from Lemma B.2. Using the assumption that the function \bar{F} follows a power law with tail exponent α , choose u_2 such that $\bar{F}(2/3r)/\bar{F}(r) \leq 2(2/3)^{-\alpha}$ for all $r > u_1$. Choose u_3 such that $\lambda c_1 r^d \bar{F}(r) \leq 1$ for all $r > u_2$. Note that $|x| \leq r_1 + r_2 \leq \frac{1}{2}|x - z|$ implies $|z| \leq |x| + |x - z| \leq \frac{3}{2}|x - z|$ and let $m = \max\{u_1, \frac{3}{2}u_2, \frac{3}{2}u_3\}$. Using Proposition 3.2 and the definition of m we have for all $|z| > m$.

$$\begin{aligned} q(|x - z|, r_1, r_2) &\leq \exp\left(\lambda \int_{\mathbb{R}_+} |B_{r_1+r}(o) \cap B_{r_2+r}(|x - z|)| F(dr)\right) - 1 \\ &\leq \exp\left(\lambda c_1 |x - z|^d \bar{F}(|x - z|)\right) - 1 \\ &\leq 2\lambda c_1 |x - z|^d \bar{F}(|x - z|). \end{aligned}$$

Note that $|x - z| \leq |x| + |z| \leq \frac{1}{2}|x - z| + |z|$ implies $|x - z| \leq 2|z|$, and that \bar{F} is a decreasing function. Now for $c = 4(2/3)^{-\alpha} 2^d \lambda c_1$ and $|z| > m$ we have

$$q(|x - z|, r_1, r_2) \leq 2\lambda c_1 (2|z|)^d \bar{F}(\frac{2}{3}|z|) \leq c|z|^d \bar{F}(|z|).$$

□

Lemma 4.3. *Fix $z \in \mathbb{R}^d$ and define*

$$A(z) = \{(x, r_1, r_2) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ : |x - z| \leq 2(r_1 + r_2)\}.$$

Then the retention covariance function q satisfies

$$\begin{aligned} \iiint_{A(z)} |B_{r_1}(o) \cap B_{r_2}(x)| |q(|x - z|, r_1, r_2)| dx F(dr_1) F(dr_2) \\ \leq 2|B_1|^2 \left(\int_{\mathbb{R}_+} r^d F(dr)\right) \left(\int_{|z|/6}^\infty r^d F(dr)\right) \sup_{r > |z|/6} h(r), \end{aligned}$$

where $h(r)$ is the weight-averaged retention probability defined by (3.3).

Proof. Define $d\mu$ as shorthand for $dx F(dr_1)F(dr_2)$, and denote the integrand by $f_z(x, r_1, r_2)$. Observe that f_z vanishes outside the set $A_0 = \{(x, r_1, r_2) : |x| < r_1 + r_2\}$. Observe also that $A(z) \cap A_0 \subset A_1(z) \cup A_2(z)$, where $A_i(z) = \{(x, r_1, r_2) : r_i > |z|/6\}$. As a consequence,

$$\int_{A(z)} f_z d\mu = \int_{A(z) \cap A_0} f_z d\mu \leq \int_{A_1(z)} f_z d\mu + \int_{A_2(z)} f_z d\mu = 2 \int_{A_1(z)} f_z d\mu,$$

where the last equality is due to the symmetry of f_z with respect to its last two arguments. Recall that $|q(|x - z|, r_1, r_2)| \leq h(r_1)$ by Lemma 3.3. Now

$$\begin{aligned} \int_{A_1(z)} f_z d\mu &\leq \iiint \mathbf{1}_{(|z|/6, \infty)}(r_1) |B_{r_1}(o) \cap B_{r_2}(x)| h(r_1) dx F(dr_1) F(dr_2) \\ &\leq J(z) \sup_{r > |z|/6} h(r), \end{aligned}$$

where

$$\begin{aligned} J(z) &= \iiint \mathbf{1}_{(|z|/6, \infty)}(r_1) |B_{r_1}(o) \cap B_{r_2}(x)| dx F(dr_1) F(dr_2) \\ &= |B_1|^2 \left(\int_{\mathbb{R}^+} r^d F(dr) \right) \left(\int_{|z|/6}^{\infty} r^d F(dr) \right). \end{aligned}$$

□

Proof of Proposition 4.1. By Proposition 3.4, we can write

$$\frac{k_{\text{th}}(z)}{\bar{F}(|z|)|z|^d} = \lambda I_1(z) + \lambda^2 (I_2(z) + I_3(z)),$$

where

$$I_1(z) = (\bar{F}(|z|)|z|^d)^{-1} \int_{\mathbb{R}^+} |B_r(o) \cap B_r(z)| h(r) F(dr),$$

and where

$$\begin{aligned} I_2(z) &= \iiint_{A_z} f_z(x, r_1, r_2) dx F(dr_1) F(dr_2), \\ I_3(z) &= \iiint_{A_z^c} f_z(x, r_1, r_2) dx F(dr_1) F(dr_2), \end{aligned}$$

denote the integrals of the function

$$f_z(x, r_1, r_2) = |B_{r_1}(o) \cap B_{r_2}(x)| \left(\frac{q(|x - z|, r_1, r_2)}{\bar{F}(|z|)|z|^d} \right)$$

over the set

$$A_z = \{(x, r_1, r_2) : |x - z| \leq 2(r_1 + r_2)\}$$

and its complement, respectively.

The integral $I_1(z) \rightarrow 0$ as $|z| \rightarrow \infty$ by Lemma B.3, because $h(r) \rightarrow 0$ as $r \rightarrow \infty$ by assumption.

We will next show that $I_2(z) \rightarrow 0$ as well. We apply Lemma 4.3, to conclude that

$$|I_2(z)| \leq c_2(|z|^d \bar{F}(|z|))^{-1} \left(\int_{|z|/6}^{\infty} r^d F(dr) \right) \sup_{r>|z|/6} h(r),$$

where $c_2 = 2|B_1|^2 \int r^d F(dr)$. The right side above tends to zero as $|z| \rightarrow \infty$, because $h(r) \rightarrow 0$ as $r \rightarrow \infty$, and because the integral on the right side above is asymptotically equivalent to constant multiple of $|z|^d \bar{F}(|z|)$ by Lemma A.1.

To analyze the limiting behavior of $I_3(z)$ as $|z| \rightarrow \infty$, note that assumption (4.1) and Lemma A.2 imply that for any x, r_1, r_2 ,

$$q(|x - z|, r_1, r_2) \sim q_\infty(r_1, r_2) |x - z|^d \bar{F}(|x - z|) \sim q_\infty(r_1, r_2) |z|^d \bar{F}(|z|).$$

By the definition of A_z , it thus follows that

$$f_z(x, r_1, r_2) \mathbf{1}_{A_z^c}(x, r_1, r_2) \rightarrow q_\infty(r_1, r_2) |B_{r_1}(o) \cap B_{r_2}(x)|$$

as $|z| \rightarrow \infty$. Moreover, by Lemma 4.2 there exists a constant c_3 such that

$$|f_z(x, r_1, r_2) \mathbf{1}_{A_z^c}(x, r_1, r_2)| \leq c_3 |B_{r_1}(o) \cap B_{r_2}(x)|$$

for all x, r_1, r_2 and all large enough z . Because the right side above is integrable with respect to $dx F(dr_1) F(dr_2)$, Lebesgue's dominated convergence theorem shows that

$$\begin{aligned} \lim_{|z| \rightarrow \infty} I_3(z) &= \iiint q_\infty(r_1, r_2) |B_{r_1}(o) \cap B_{r_2}(x)| dx F(dr_1) F(dr_2) \\ &= |B_1|^2 \iint q_\infty(r_1, r_2) r_1^d r_2^d F(dr_1) F(dr_2), \end{aligned}$$

which completes the proof of Proposition 4.1. \square

4.2 Asymptotic two-point correlation

Proposition 4.4. *Assume that the radius distribution F follows a power law with tail exponent $\alpha > d$ and*

$$q(|z|, r_1, r_2) \sim q_\infty(r_1, r_2) |z|^d \bar{F}(|z|) \quad \text{as } |z| \rightarrow \infty. \quad (4.2)$$

Then

$$\xi_{\text{th}}(z) \sim c |z|^d \bar{F}(|z|) \quad \text{as } |z| \rightarrow \infty,$$

where

$$c = \frac{\lambda^2}{\lambda_{\text{th}}^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} q_\infty(r_1, r_2) F(dr_1) F(dr_2).$$

Proof. Using (3.14) we can write

$$\frac{\xi_{\text{th}}(z)}{|z|^d \bar{F}(|z|)} = \frac{\lambda^2}{\lambda_{\text{th}}^2} (I_1(z) + I_2(z)),$$

where

$$\begin{aligned} I_1(z) &= \iint_{A_z} f_z(r_1, r_2) F(dr_1) F(dr_2), \\ I_2(z) &= \iint_{A_z^c} f_z(r_1, r_2) F(dr_1) F(dr_2), \end{aligned}$$

denote the integrals of

$$f_z(r_1, r_2) = \frac{q(|z|, r_1, r_2)}{|z|^d \bar{F}(|z|)}$$

over the set

$$A_z = \{(r_1, r_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : r_1 + r_2 > |z|/2\}$$

and its complement, respectively.

Observe that $A_z \subset A_1(z) \cup A_2(z)$, where $A_i(z) = \{(r_1, r_2) : r_i > |z|/4\}$, and that $|q| \leq 1$ by Lemma 3.3. As a consequence,

$$\int_{A_z} |q(|z|, r_1, r_2)| F(dr_1) F(dr_2) \leq (F \times F)(A_1(z)) + (F \times F)(A_2(z)) = 2\bar{F}(|z|/4),$$

which implies $I_1(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

Note that $f_z(r_1, r_2) \mathbf{1}_{A_z^c}(r_1, r_2) \rightarrow q_\infty(r_1, r_2)$ by assumption (4.2) and the definition of A_z . By Lemma 4.2, $f_z(r_1, r_2) \mathbf{1}_{A_z^c}(r_1, r_2)$ is bounded for large z uniformly on r_1 and r_2 . Lebesgue's dominated convergence theorem then shows that

$$\lim_{|z| \rightarrow \infty} I_2(z) = \iint q_\infty(r_1, r_2) F(dr) F(ds).$$

□

5 Isolated grains retained

In this section we study the thinning where only isolated grains are retained. In the general framework of Section 3, this is achieved by assigning unit weight to every grain, so that $G_r(dw) = \delta_1(dw)$. For nonrandom equally sized grains this corresponds to the classical Matérn type I thinning.

Theorem 5.1. *Assume that the radius distribution F follows a power law with tail exponent $\alpha > d$, so that $1 - F(r) = \ell(r)r^{-\alpha}$ for some slowly varying function ℓ . Then the thinned radius distribution is bounded by*

$$\bar{F}_{\text{th}}(r) \leq \frac{\lambda}{\lambda_{\text{th}}} e^{-\lambda|B_1|r^d},$$

the covariance function of the thinned grain cover decays according to

$$k_{\text{th}}(z) \sim \lambda c_{\alpha,d} p_{\text{th}}^2 \ell(|z|) |z|^{-(\alpha-d)} \quad \text{as } |z| \rightarrow \infty,$$

and the two-point correlation function of the thinned grain centers according to

$$\xi_{\text{th}}(z) \sim \lambda c_{\alpha,d} \ell(|z|) |z|^{-(\alpha-d)} \quad \text{as } |z| \rightarrow \infty,$$

where the constant $c_{\alpha,d}$ is given by (B.2).

Proof. Because the weights are deterministic the retention probabilities have simple formulas

$$h(r) = h(r, 1) = \exp \left(-\lambda |B_1| \int_{\mathbb{R}_+} (r+s)^d F(ds) \right)$$

and

$$h_2(|z|, r_1, r_2) = h(r_1)h(r_2) \exp \left(\lambda \int_{\mathbb{R}_+} |B_{r_1+s}(o) \cap B_{r_2+s}(z)| F(ds) \right).$$

The tail of the thinned radius distribution (3.6) is

$$\bar{F}_{\text{th}}(r) = \frac{\lambda}{\lambda_{\text{th}}} \int_r^\infty h(s) F(ds) \leq \frac{\lambda}{\lambda_{\text{th}}} h(r) \leq \frac{\lambda}{\lambda_{\text{th}}} e^{-\lambda |B_1| r^d}.$$

To show the claim for the covariance and two-point correlation functions, we will use Proposition 4.1 and Proposition 4.4 respectively. For that we need to show that (4.1) holds. By Lemma B.1 we have for the average intersection volume in h_2 above

$$\int_{\mathbb{R}_+} |B_{r_1+s}(o) \cap B_{r_2+s}(z)| F(ds) \sim c |z|^d \bar{F}(|z|).$$

Because the right hand side goes to zero as $|z| \rightarrow \infty$ we can use the fact that $\lim_{t \rightarrow \infty} (e^t - 1)/t = 1$ to obtain (4.1) with

$$q_\infty(r_1, r_2) = h(r_1)h(r_2)\lambda c_{\alpha,d}.$$

Using Proposition 4.1 we find that $k_{\text{th}}(z) \sim c_1 |z|^d \bar{F}(|z|)$. Using the formula for volume fraction (3.7) we also find the constant $c_1 = \lambda p_{\text{th}}^2 c_{\alpha,d}$. Similarly by Proposition 4.4 we find that $\xi_{\text{th}}(z) \sim c_2 |z|^d \bar{F}(|z|)$. With the help of germ density (3.5) we have $c_2 = \lambda c_{\alpha,d}$. \square

6 Random grains retained

Here we assume that each grain in the proposed Boolean model is assigned a random weight independently of the other grains, according to some continuous distribution function. The continuity ensures that there will be no tie breaks. Because the shape of the weight distribution does not affect the retention probabilities considered here, as long as it is continuous, we may without loss of generality assume that $G_r(dw) = 1_{(0,1)}(w)dw$, the uniform distribution on $(0, 1)$. Note that for nonrandom equally sized grains, this corresponds to the classical Matérn type II thinning.

Theorem 6.1. *Assume that the radius distribution F follows a power law with tail exponent $\alpha > d$, so that $1 - F(r) = \ell(r)r^{-\alpha}$ for some slowly varying function ℓ . Then the thinned radius distribution decays according to*

$$\bar{F}_{\text{th}}(r) \sim (\lambda_{\text{th}}|B_1|)^{-1} \frac{\alpha}{\alpha + d} \ell(r) r^{-(\alpha+d)} \quad \text{as } r \rightarrow \infty,$$

the covariance function of the thinned grain cover according to

$$k_{\text{th}}(z) \sim c_1 \ell(|z|) |z|^{-(\alpha-d)} \quad \text{as } |z| \rightarrow \infty,$$

and the two-point correlation function of the thinned grain centers according to

$$\xi_{\text{th}}(z) \sim c_2 \ell(|z|) |z|^{-(\alpha-d)} \quad \text{as } |z| \rightarrow \infty,$$

for some $c_1, c_2 \in (0, \infty)$.

Proof. The retention probability of a grain with radius r and weight $w \in (0, 1)$ is

$$h(r, w) = \exp \left(-\lambda \int_{\mathbb{R}_+} G_R[w, \infty) |B_{r+s}(o)| F(ds) \right) = \exp(-\lambda(1-w)b(r)),$$

where

$$b(r) = \int_{\mathbb{R}_+} |B_{r+s}(o)| F(ds).$$

The weight-averaged retention probability thus equals

$$h(r) = \int_0^1 e^{-\lambda(1-w)b(r)} dw = \frac{1 - e^{-\lambda b(r)}}{\lambda b(r)}.$$

Because $b(r) \rightarrow \infty$ as $r \rightarrow \infty$, it follows that $\lim_{r \rightarrow \infty} h(r) = 0$. The first condition of Proposition 4.1 is thus satisfied.

Note that $b(r) \sim |B_r|$ as $r \rightarrow \infty$, which implies that

$$h(r) \sim (\lambda b(r))^{-1} \sim (\lambda |B_1| r^d)^{-1}.$$

By Lemma A.4, the thinned radius distribution (3.6) is

$$\bar{F}_{\text{th}}(r) = \frac{\lambda}{\lambda_{\text{th}}} \int_r^\infty h(s) F(ds) \sim (\lambda_{\text{th}} |B_1|)^{-1} \int_r^\infty s^{-d} F(ds).$$

Furthermore by Lemma A.1

$$\bar{F}_{\text{th}}(r) \sim (\lambda_{\text{th}} |B_1|)^{-1} \frac{\alpha}{\alpha + d} r^{-d} \bar{F}(r).$$

The pair retention probability equals

$$\begin{aligned} & h_2(|z|, r_1, r_2, w_1, w_2) \\ &= h(r_1, w_1) h(r_2, w_2) \exp \left(\lambda \int_{\mathbb{R}_+} G_R[w_1 \vee w_2, \infty) |B_{r_1+s}(o) \cap B_{r_2+s}(z)| F(ds) \right) \\ &= \exp \left(-\lambda(1-w_1)b(r_1) - \lambda(1-w_2)b(r_2) + \lambda(1-w_1 \vee w_2)a_z(r_1, r_2) \right), \end{aligned}$$

where

$$a_z(r_1, r_2) = \int_{\mathbb{R}^+} |B_{r_1+s}(o) \cap B_{r_2+s}(z)| F(ds).$$

From this expression we see that the retention covariance function defined in (3.11) equals

$$q(|z|, r_1, r_2) = \int_0^1 \int_0^1 e^{-\lambda b(r_1)(1-w_1)} e^{-\lambda b(r_2)(1-w_2)} \left(e^{\lambda(1-w_1 \vee w_2)a_z(r_1, r_2)} - 1 \right) dw_1 dw_2.$$

As $|z| \rightarrow \infty$, Lemma B.1 shows that the term in parentheses above is asymptotically equivalent to

$$e^{\lambda(1-w_1 \vee w_2)a_z(r_1, r_2)} - 1 \sim \lambda(1-w_1 \vee w_2)c_{\alpha, d}\bar{F}(|z|)|z|^d.$$

With the help of the bound $|e^t - 1| \leq (e-1)t$ for $t \in [0, 1]$, we may use dominated convergence to conclude that

$$q(|z|, r_1, r_2) \sim q_\infty(r_1, r_2)|z|^d\bar{F}(|z|) \quad \text{as } |z| \rightarrow \infty,$$

where

$$q_\infty(r_1, r_2) = \lambda c_{\alpha, d} \int_0^1 \int_0^1 (1-w_1 \vee w_2) e^{-\lambda b(r_1)(1-w_1)} e^{-\lambda b(r_2)(1-w_2)} dw_1 dw_2.$$

Now by Proposition 4.1 it follows that

$$k_{\text{th}}(z) \sim c_1 \bar{F}(|z|)|z|^d \quad \text{as } |z| \rightarrow \infty,$$

where

$$c_1 = \lambda^2 |B_1|^2 \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r_1^d r_2^d q_\infty(r_1, r_2) F(dr_1) F(dr_2).$$

The constant c_1 is finite because $q_\infty(r_1, r_2) \leq \lambda c_{\alpha, d}$ for all r_1, r_2 . The fact that c_1 is strictly positive is easily seen by inspecting the expression of $q_\infty(r_1, r_2)$.

Similarly, Proposition 4.4 shows that

$$\xi_{\text{th}}(z) \sim c_2 |z|^d \bar{F}(|z|),$$

where

$$c_2 = \frac{\lambda^2}{\lambda_{\text{th}}^2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} q_\infty(r_1, r_2) F(dr_1) F(dr_2).$$

The finiteness and strict positivity of c_2 follow by similar reasoning as for c_1 . \square

7 Large grains retained

A thinning which favors large grains is obtained by letting the weight of each grain be equal to its radius, so that $G_r(dw) = \delta_r(dw)$.

Theorem 7.1. *Assume that the radius distribution F follows a power law with tail exponent $\alpha > d$, so that $1 - F(r) = \ell(r)r^{-\alpha}$ for some slowly varying function ℓ . Then the thinned radius distribution decays according to*

$$\bar{F}_{\text{th}}(r) \sim \frac{\lambda}{\lambda_{\text{th}}} \ell(r)r^{-\alpha} \quad \text{as } r \rightarrow \infty,$$

the covariance function of the thinned grain cover according to

$$k_{\text{th}}(z) \sim \lambda c_{\alpha,d} (1 - p_{\text{th}})^2 \ell(|z|) |z|^{-(\alpha-d)} \quad \text{as } |z| \rightarrow \infty,$$

and the two-point correlation function of the thinned grain centers according to

$$\xi_{\text{th}}(z) \sim \lambda c_{\alpha,d} \ell(|z|) |z|^{-(\alpha-d)} \quad \text{as } |z| \rightarrow \infty,$$

where the constant $c_{\alpha,d}$ is given by (B.2).

Proof. Because the weight of each grain is equal to its radius, the weight-averaged retention probability $h(r)$ is equal to $h(r, w)$ with w taking on the value r . By Proposition 3.1, the retention probability is given by

$$h(r) = \exp \left(-\lambda \int_{\mathbb{R}_+} |B_{r+s}| \mathbf{1}_{[r,\infty)}(s) F(ds) \right).$$

Because the integrand above tends to zero as $r \rightarrow \infty$, and the integrand is bounded by the $F(ds)$ -integrable function $|B_{2s}|$, dominated convergence implies that $\lim_{r \rightarrow \infty} h(r) = 1$. By Lemma A.4, the tail of the thinned radius distribution (3.6) satisfies

$$\bar{F}_{\text{th}}(r) = \frac{\lambda}{\lambda_{\text{th}}} \int_r^\infty h(s) F(ds) \sim \frac{\lambda}{\lambda_{\text{th}}} \int_r^\infty F(ds) = \frac{\lambda}{\lambda_{\text{th}}} \bar{F}(r).$$

To analyze the long-range behavior of $k_{\text{th}}(z)$ and $\xi_{\text{th}}(z)$, let us first investigate the long-range behavior of the retention covariance function $q(|z|, r_1, r_2)$ defined by (3.11). Using Proposition 3.2, we find that

$$q(|z|, r_1, r_2) = h(r_1)h(r_2) \left(\mathbf{1}(r_1 + r_2 < |z|) e^{\tau(|z|, r_1, r_2)} - 1 \right), \quad (7.1)$$

where

$$\tau(|z|, r_1, r_2) = \lambda \int_{\mathbb{R}_+} \mathbf{1}_{[r_1 \vee r_2, \infty)}(s) |B_{r_1+s}(o) \cap B_{r_2+s}(z)| F(ds). \quad (7.2)$$

When $|z| > 3(r_1 + r_2)$, we may replace the region of integration above with the full positive real line, so that with the help of Lemma B.1 we find that

$$\tau(|z|, r_1, r_2) = \lambda \int_{\mathbb{R}_+} |B_{r_1+s}(o) \cap B_{r_2+s}(z)| F(ds) \sim \lambda c_{\alpha,d} |z|^d \bar{F}(|z|),$$

as $|z| \rightarrow \infty$. Because $e^t - 1 \sim t$ for small t , we conclude using (7.1) that

$$q(|z|, r_1, r_2) \sim q_\infty(r_1, r_2)|z|^d \bar{F}(|z|), \quad (7.3)$$

where

$$q_\infty(r_1, r_2) = \lambda c_{\alpha, d} h(r_1) h(r_2). \quad (7.4)$$

The claim for the two-point correlation function $\xi_{\text{th}}(z)$ now follows by using Proposition 4.4, after noting that the constant in Proposition 4.4 is

$$\frac{\lambda^2}{\lambda_{\text{th}}^2} \iint q_\infty(r_1, r_2) F(dr_1) F(dr_2) = \lambda \lambda_{\text{th}}^{-2} c_{\alpha, d} \left(\lambda \int h(r) F(dr) \right)^2 = \lambda c_{\alpha, d}.$$

We will now move on to the part concerning the covariance function $k_{\text{th}}(z)$ of the thinned grain cover. Note that because $h(r)$ does not vanish as $r \rightarrow \infty$, we cannot use Proposition 4.1 to deduce the long-range behavior of $k_{\text{th}}(z)$. Instead, we will proceed by directly analyzing the integral building blocks of $k_{\text{th}}(z)$ in high precision. Let us start by rewriting (3.13) as

$$k_{\text{th}}(z) = \lambda I_0(z) + \lambda^2 (I_1(z) + I_2(z) + I_3(z)),$$

where

$$I_0(z) = \int_{\mathbb{R}_+} |B_r(o) \cap B_r(z)| h(r) F(dr),$$

$$I_j(z) = \iint_{A_j^z} |B_{r_1}(o) \cap B_{r_2}(x)| q(|x - z|, r_1, r_2) dx F(dr_1) F(dr_2), \quad j = 1, 2, 3,$$

and

$$\begin{aligned} A_1^z &= \{(x, r_1, r_2) : |x| < r_1 + r_2, r_1 + r_2 < |x - z|/2\}, \\ A_2^z &= \{(x, r_1, r_2) : |x| < r_1 + r_2, |x - z|/2 < r_1 + r_2 < |x - z|\}, \\ A_3^z &= \{(x, r_1, r_2) : |x| < r_1 + r_2, |x - z| < r_1 + r_2\}. \end{aligned}$$

The first term $I_0(z) \lesssim c_{\alpha, d} |z|^d \bar{F}(|z|)$ by Lemma B.1. Note that the integrand in $I_0(z)$ vanishes for $r \leq |z|/2$ so that

$$I_0(z) \geq \inf_{r \geq |z|/2} h(r) \int_{\mathbb{R}_+} |B_r(o) \cap B_r(z)| F(dr).$$

Using Lemma B.1 and that $h(r) \rightarrow 1$ as $r \rightarrow \infty$ we conclude that

$$I_0(z) \sim c_{\alpha, d} |z|^d \bar{F}(|z|). \quad (7.5)$$

Next, we will prove that

$$I_1(z) \sim \lambda^{-1} c_{\alpha, d} p_{\text{th}}^2 |z|^d \bar{F}(|z|). \quad (7.6)$$

By Lemma 4.2, the function $(x, r_1, r_2) \mapsto \frac{q(|x-z|, r_1, r_2)}{|z|^d \bar{F}(|z|)} \mathbf{1}_{A_1^z}(x, r_1, r_2)$ is positive and bounded by a constant which does not depend on z . Because $|B_{r_1}(o) \cap B_{r_2}(x)|$ is integrable with respect to $dx F(dr_1) F(dr_2)$, Lebesgue's dominated convergence theorem shows that

$$\lim_{z \rightarrow \infty} \frac{I_1(z)}{|z|^d \bar{F}(|z|)} = \iiint |B_{r_1}(o) \cap B_{r_2}(x)| \left(\lim_{z \rightarrow \infty} \frac{q(|x-z|, r_1, r_2)}{|z|^d \bar{F}(|z|)} \mathbf{1}_{A_1^z} \right) dx F(dr_1) F(dr_2).$$

Using (7.3) and the definition of A_1^z , the limit on the right equals $q_\infty(r_1, r_2)$. Plugging in the expression (7.4) for $q_\infty(r_1, r_2)$ and recalling the formula (3.7) for the volume fraction of the thinned grain cover p_{th} , we find that

$$\begin{aligned} \lim_{|z| \rightarrow \infty} \frac{I_1(z)}{|z|^d \bar{F}(|z|)} &= \lambda c_{\alpha, d} \iiint |B_{r_1}(o) \cap B_{r_2}(x)| h(r_1) h(r_2) dx F(dr_1) F(dr_2) \\ &= \lambda c_{\alpha, d} \iint |B_1|^2 r_1^d r_2^d h(r_1) h(r_2) F(dr_1) F(dr_2) \\ &= \lambda^{-1} c_{\alpha, d} p_{\text{th}}^2, \end{aligned}$$

which proves the validity of (7.6).

Now we will prove that

$$\frac{I_2(z)}{|z|^d \bar{F}(|z|)} \rightarrow 0 \quad \text{as } |z| \rightarrow \infty. \quad (7.7)$$

First, using the bound $|B_{r_1+s}(o) \cap B_{r_2+s}(x-z)| \leq |B_{r_1+s}| \leq |B_{2s}|$ for $s \geq r_1 \vee r_2$, we find that the function τ defined in (7.2) is bounded by

$$\tau(|x-z|, r_1, r_2) \leq \lambda |B_1| 2^d \int \mathbf{1}_{[r_1 \vee r_2, \infty)}(s) s^d F(ds).$$

Observe next that

$$|z| \leq 3(r_1 + r_2) \leq 6(r_1 \vee r_2)$$

for all $(x, r_1, r_2) \in A_2^z$, so that $r_1 \vee r_2$ is large when $|z|$ is large. As a consequence, we see by Lemma A.1 and Lemma A.3 that for all $(x, r_1, r_2) \in A_2^z$ and all large enough z ,

$$\begin{aligned} \tau(|x-z|, r_1, r_2) &\leq 2\lambda |B_1| 2^d (r_1 \vee r_2)^d \bar{F}(r_1 \vee r_2) \\ &\leq 4\lambda |B_1| 2^d (|z|/6)^d \bar{F}(|z|/6). \end{aligned}$$

Because $e^t - 1 \leq (e-1)t$ for $t \in [0, 1]$, formula (7.1) combined with the above inequality shows that for all $(x, r_1, r_2) \in A_2^z$ and all large enough z ,

$$0 \leq q(|x-z|, r_1, r_2) \leq c_1 |z|^d \bar{F}(|z|/6),$$

where $c_1 = 4(e-1)\lambda |B_1| 3^{-d}$. Therefore,

$$0 \leq I_2(z) \leq c_1 |z|^d \bar{F}(|z|/6) \iiint_{A_2^z} |B_{r_1}(o) \cap B_{r_2}(x)| dx F(dr_1) F(dr_2).$$

Note that $A_2^z \subset A_{21}^z \cup A_{22}^z$ where $A_{2i}^z = \{(x, r_1, r_2) : r_i \geq |z|/6\}$, $i = 1, 2$. By symmetry of the integrand with respect to r_1 and r_2 ,

$$\begin{aligned} 0 \leq I_2(z) &\leq 2c_1|z|^d \bar{F}(|z|/6) \int_{\mathbb{R}_+} \int_{|z|/6}^{\infty} \int_{\mathbb{R}^d} |B_{r_1}(o) \cap B_{r_2}(x)| dx F(dr_1) F(dr_2) \\ &= 2c_1|z|^d \bar{F}(|z|/6) |B_1|^2 \left(\int r^d F(dr) \right) \left(\int_{|z|/6}^{\infty} r^d F(dr) \right), \end{aligned}$$

which shows the validity of (7.7).

It remains to be shown that

$$I_3(z) \sim -2 \frac{p_{\text{th}}}{\lambda} c_{\alpha, d} |z|^d \bar{F}(|z|). \quad (7.8)$$

To do that, we first fix a small $\epsilon \in (0, 1/4)$. Note that by formula (7.1), the retention covariance function equals $q(|x - z|, r_1, r_2) = -h(r_1)h(r_2)$ for $(x, r_1, r_2) \in A_3^z$. Note also that, for fixed r_1 and r_2 the x -slice of A_3^z is

$$\{x : (x, r_1, r_2) \in A_3^z\} = B_{r_1+r_2}(z) \cap B_{r_1+r_2}(o).$$

Because $|B_{r_1}(o) \cap B_{r_2}(x)|$ vanishes for x outside $B_{r_1+r_2}(o)$, we may represent $I_3(z)$ according to

$$\begin{aligned} I_3(z) &= - \iiint_{A_3^z} |B_{r_1}(o) \cap B_{r_2}(x)| h(r_1)h(r_2) dx F(dr_1) F(dr_2) \\ &= - \iint_{C_3^z} \int_{B_{r_1+r_2}(z)} |B_{r_1}(o) \cap B_{r_2}(x)| dx h(r_1)h(r_2) F(dr_1) F(dr_2) \end{aligned}$$

where

$$C_3^z = \{(r_1, r_2) : |z| \leq 2(r_1 + r_2)\}.$$

Next we split $I_3(z)$ into three parts

$$I_3(z) = -(I_{31}(z) + I_{32}(z) + I_{33}(z)),$$

where

$$I_{3j}(z) = \iint_{A_{3j}^z} \int_{B_{r_1+r_2}(z)} |B_{r_1}(o) \cap B_{r_2}(x)| dx h(r_1)h(r_2) F(dr_1) F(dr_2)$$

for $j = 1, 2, 3$ and

$$\begin{aligned} C_{31}^z &= C_{311}^z \cup C_{312}^z \\ C_{311}^z &= \{(r_1, r_2) : 0 \leq r_1 \leq \epsilon|z|, |z|/2 \leq r_2\} \\ C_{312}^z &= \{(r_1, r_2) : 0 \leq r_2 \leq \epsilon|z|, |z|/2 \leq r_1\} \\ C_{32}^z &= C_{331}^z \cup C_{332}^z \\ C_{321}^z &= \{(r_1, r_2) : 0 \leq r_1 \leq \epsilon|z|, |z|/2 - r_1 \leq r_2 \leq |z|/2\} \\ C_{322}^z &= \{(r_1, r_2) : 0 \leq r_2 \leq \epsilon|z|, |z|/2 - r_2 \leq r_1 \leq |z|/2\} \\ C_{33}^z &= [\epsilon|z|, \infty)^2 \cap C_3^z. \end{aligned}$$

A change of variables shows that

$$\int_{B_{r_1+r_2}(z)} |B_{r_1}(o) \cap B_{r_2}(x)| dx = \int_{B_{r_1}(o)} |B_{r_2}(z) \cap B_{r_1+r_2}(x)| dx,$$

so that we can express the integral $I_{31}(z)$ more conveniently as

$$I_{31}(z) = \iint_{C_{31}^z} \int_{B_{r_1}(o)} |B_{r_2}(z) \cap B_{r_1+r_2}(x)| dx h(r_1)h(r_2) F(dr_1)F(dr_2).$$

By symmetry we can write $I_{31}(z) = 2I_{311}(z)$, where $I_{311}(z)$ is a modification of $I_{31}(z)$ with the region of integration C_{31}^z replaced by C_{311}^z . To analyze the long-range behavior of $I_{311}(z)$, let us split it according to $I_{311}(z) = J_1(z) + J_2(z)$, where

$$J_1(z) = \int_0^{\epsilon|z|} \int_{|z|/2}^{\infty} \int_{B_{r_1}(o)} |B_{r_2}(z) \cap B_{r_2}(o)| dx h(r_2) F(dr_2) h(r_1) F(dr_1),$$

and where $J_2(z) = I_{311}(z) - J_1(z)$. Because the integrand of $J_1(z)$ does not depend on x , we can rewrite the integral as

$$J_1(z) = \left(\int_0^{\epsilon|z|} |B_r| h(r) F(dr) \right) \left(\int_{|z|/2}^{\infty} |B_r(z) \cap B_r(o)| h(r) F(dr) \right).$$

The first integral on the right satisfies

$$\int_0^{|z|\epsilon} |B_1| r^d h(r) F(dr) \sim \int_{\mathbb{R}_+} |B_1| r^d h(r) F(dr) = \lambda^{-1} p_{\text{th}},$$

where p_{th} is the volume fraction of the thinned grain cover given by (3.7). Notice that, because the intersection in the second integral vanishes for $r < |z|/2$, we can apply (7.5) to conclude that

$$J_1(z) \sim \lambda^{-1} p_{\text{th}} c_{\alpha,d} |z|^d \bar{F}(|z|).$$

The rest of the proof constitutes of showing that the remaining three parts of $I_3(z)$ are negligible. We start by showing that $J_2 \geq 0$ and

$$\limsup_{|z| \rightarrow \infty} \frac{J_2(z)}{|z|^d \bar{F}(|z|)} \leq |B_1|^2 \left((1 + 2\epsilon)^d - 1 \right) \left(\int r^d F(dr) \right) \frac{\alpha}{\alpha - d} 2^{\alpha-d}.$$

First we need a bound for the difference of the intersections in $J_2(z)$. Fix $x \in B_{r_1}(o)$ and $(r_1, r_2) \in C_{311}^z$. Because $|x| \leq r_1$, we have $B_{r_2}(o) \subset B_{r_1+r_2}(x)$, which implies that the integrand in $J_2(z)$ is bounded by

$$\begin{aligned} 0 &\leq |B_{r_2}(z) \cap B_{r_1+r_2}(x)| - |B_{r_2}(z) \cap B_{r_2}(o)| \\ &= |B_{r_2}(z) \cap (B_{r_1+r_2}(x) \setminus B_{r_2}(o))| \\ &\leq |B_{r_1+r_2}(x)| - |B_{r_2}(o)| \\ &= |B_1| ((1 + r_1/r_2)^d - 1) r_2^d \\ &\leq |B_1| ((1 + 2\epsilon)^d - 1) r_2^d, \end{aligned}$$

where the last inequality is due to $r_1 \leq \epsilon|z|$ and $|z|/2 \leq r_2$. This bound and $h(r) \leq 1$ now imply that

$$\begin{aligned} 0 \leq J_2(z) &\leq \int_0^{\epsilon|z|} \int_{|z|/2}^{\infty} \int_{B_{r_1}(o)} |B_1|((1+2\epsilon)^d - 1) r_2^d dx h(r_2) F(dr_2) h(r_1) F(dr_1) \\ &\leq |B_1|^2 ((1+2\epsilon)^d - 1) \left(\int_0^{\epsilon|z|} r^d F(dr) \right) \left(\int_{|z|/2}^{\infty} r^d F(dr) \right). \end{aligned}$$

Now using Lemma A.1 proves the claim.

We will now show that $I_{32} \geq 0$ and

$$\limsup_{|z| \rightarrow \infty} \frac{I_{32}(z)}{|z|^d \bar{F}(|z|)} \leq 2 \left(\int r^d F(dr) \right) |B_1|^2 \left[(1-2\epsilon)^{d-\alpha} - 1 \right] \frac{\alpha}{\alpha-d} 2^{\alpha-d}.$$

By symmetry $I_{32}(z) = 2I_{321}(z)$, where

$$I_{321}(z) = \iint_{C_{321}^z} \int_{B_{r_1+r_2}(z)} |B_{r_1}(o) \cap B_{r_2}(x)| dx h(r_1) h(r_2) F(dr_1) F(dr_2).$$

Note that $C_{321}^z \subset [0, \epsilon|z|] \times [|z|/2(1-2\epsilon), |z|/2]$. Also approximating $B_{r_1+r_2}(z)$ by \mathbb{R}^d and recalling that $h(r) \leq 1$ we have

$$\begin{aligned} I_{321}(z) &\leq \int_0^{|z|\epsilon} \int_{(|z|/2)(1-2\epsilon)}^{|z|/2} |B_1|^2 r_1^d r_2^d F(dr_2) F(dr_1) \\ &= |B_1|^2 \left(\int_0^{|z|\epsilon} r^d F(dr) \right) \left(\int_{(|z|/2)(1-2\epsilon)}^{\infty} r^d F(dr) - \int_{|z|/2}^{\infty} r^d F(dr) \right). \end{aligned}$$

Now using Lemma A.1 implies the claim.

For the last part $I_{33}(z)$ we have first a simple bound

$$\begin{aligned} I_{33}(z) &\leq \int_{\epsilon|z|}^{\infty} \int_{\epsilon|z|}^{\infty} \int_{B_{r_1+r_2}(z)} |B_{r_1}(o) \cap B_{r_2}(x)| dx h(r_1) h(r_2) F(dr_1) F(dr_2) \\ &\leq \int_{\epsilon|z|}^{\infty} \int_{\epsilon|z|}^{\infty} |B_1|^2 r_1^d r_2^d F(dr_1) F(dr_2) \\ &= |B_1|^2 \left(\int_{\epsilon|z|}^{\infty} r^d F(dr) \right)^2. \end{aligned}$$

Using Lemma A.1 this bound implies that

$$\frac{I_{33}(z)}{|z|^d \bar{F}(z)} \rightarrow 0 \quad \text{as } |z| \rightarrow \infty.$$

Adding together all the parts of $I_3(z)$ we have

$$\limsup_{|z| \rightarrow \infty} \frac{I_3(z)}{|z|^d \bar{F}(|z|)} = -2 \frac{p_{\text{th}}}{\lambda} c_{\alpha,d}$$

and

$$\liminf_{|z| \rightarrow \infty} \frac{I_3(z)}{|z|^d \bar{F}(|z|)} \geq -2 \frac{p_{\text{th}}}{\lambda} c_{\alpha,d} - \delta(\epsilon),$$

where

$$\delta(\epsilon) = 2|B_1|^2 \left(\int r^d F(dr) \right) \frac{\alpha}{\alpha - d} 2^{\alpha-d} \left(\left((1+2\epsilon)^d - 1 \right) + \left((1-2\epsilon)^{d-\alpha} - 1 \right) \right).$$

Letting $\epsilon \rightarrow 0$ shows the validity of (7.8) and concludes the proof. \square

8 Small grains retained

In this section we study a thinning which favors small grains. This thinning is obtained by setting the weight of each grain to the inverse of its radius, so that $G_r(dw) = \delta_{1/r}(dw)$. The following theorem shows that the thinned radius distribution and the key second-order characteristics decay rapidly to zero, regardless of the tail behavior of the original radius distribution F . Note that here, unlike in Theorems 5.1–7.1, there is no need to assume anything on the shape of the radius distribution F .

Theorem 8.1. *Assume that the radius distribution F satisfies $\int r^d F(dr) < \infty$. Then the thinned radius distribution is bounded by*

$$\bar{F}_{\text{th}}(r) \leq \frac{\lambda}{\lambda_{\text{th}}} e^{-\lambda|B_1|\frac{1}{2}r^d},$$

the covariance function of the thinned grain cover by

$$|k_{\text{th}}(z)| \leq e^{-\lambda|B_1|c|z|^d},$$

and the two-point correlation function of the thinned grain centers by

$$|\xi_{\text{th}}(z)| \leq e^{-\lambda|B_1|c|z|^d}$$

for all large values of r and z , where $c \in (0, \infty)$.

Proof. Now the weight-averaged retention probability $h(r)$ is equal to the retention probability of a reference grain with radius r and weight $1/r$. Using Proposition 3.1 we find that

$$h(r) = \exp \left(-\lambda \int_0^r |B_{r+s}(o)| F(ds) \right).$$

From this expression we see that h decreases monotonically to zero as r grows, and that $h(r) \leq \exp(-\frac{1}{2}\lambda|B_1|r^d)$ for all large enough r so that $F(r) \geq 1/2$. Proposition 3.2 further shows that the weight-averaged pair retention probability equals

$$h_2(|z|, r_1, r_2) = h(r_1)h(r_2) \exp \left(\lambda \int \mathbf{1}_{[0, r_1 \wedge r_2]}(r) |B_{r_1+r}(o) \cap B_{r_2+r}(z)| F(dr) \right) \quad (8.1)$$

for $|z| > r_1 + r_2$.

To analyze the covariance function of the thinned grain cover, recall that

$$\begin{aligned} k_{\text{th}}(z) &= \lambda \int |B_r(o) \cap B_r(z)| h(r) F(dr) \\ &\quad + \lambda^2 \iint |B_{r_1}(o) \cap B_{r_2}(x)| q(|x - z|, r_1, r_2) dxF(dr_1) F(dr_2), \end{aligned} \quad (8.2)$$

where $q(u, r_1, r_2) = h_2(u, r_1, r_2) - h(r_1)h(r_2)$. Because $|B_r(o) \cap B_r(z)| \leq |B_1|r^d \mathbf{1}(r > |z|/2)$, the first term on the right side of (8.2) is bounded from above by

$$\lambda|B_1| \left(\int_{|z|/2}^{\infty} r^d F(dr) \right) \sup_{r > |z|/2} h(r).$$

Note that $q(|x - z|, r_1, r_2)$ vanishes for $|x - z| > 2(r_1 + r_2)$, because the integral in (8.1) vanishes for $|z| \geq 2(r_1 + r_2)$. This is why the integration in second term in (8.2) can be restricted to the set $A(z) = \{(x, r_1, r_2) : |x - z| \leq 2(r_1 + r_2)\}$. Now using Lemma 4.3, the absolute value of second term in (8.2) is bounded from above by

$$2\lambda^2|B_1|^2 \left(\int_{\mathbb{R}^+} r^d F(dr) \right) \left(\int_{|z|/6}^{\infty} r^d F(dr) \right) \sup_{r > |z|/6} h(r).$$

As consequence, we find that

$$|k_{\text{th}}(z)| \leq (\lambda m_1 + 2\lambda^2 m_1^2) \sup_{r > |z|/6} h(r),$$

where $m_1 = |B_1| \int r^d F(dr)$ is the mean volume of a grain. Therefore,

$$|k_{\text{th}}(z)| \leq (\lambda m_1 + 2\lambda^2 m_1^2) e^{-\frac{1}{2}\lambda|B_1|(|z|/6)^d}$$

for all large enough z such that $F(|z|) \geq 1/2$. A similar analysis can be carried out for the two-point correlation function.

Typical radius has tail probabilities (3.6)

$$\bar{F}_{\text{th}}(r) = \frac{\lambda}{\lambda_{\text{th}}} \int_r^{\infty} h(s) F(ds) \leq \frac{\lambda}{\lambda_{\text{th}}} h(r) \bar{F}(r) \leq \frac{\lambda}{\lambda_{\text{th}}} e^{-\frac{1}{2}\lambda|B_1|r^d}$$

for all large enough r so that $F(r) \geq 1/2$. \square

9 Conclusions and future work

Boolean models consisting of randomly sized spheres in \mathbb{R}^d are long-range dependent if the sphere radii follow a power-law distribution with tail exponent $\alpha \in (d, 2d)$. We studied second-order statistical properties of four hard-core germ-grain models

which are obtained from such Boolean models using a natural weight-based thinning mechanism. We found that a thinning which favors large grains preserves the power-law covariance decay of the proposed Boolean model, whereas a thinning which favors small grains does not. The power-law covariance decay is also preserved under the thinning where only isolated grains are retained (Matérn type I), and the thinning where retention is determined by independent weights (Matérn type II). The germ–grain model obtained by the Matérn type I thinning is an interesting example of a homogeneous hard-sphere model where typical spheres have exponentially small sizes but the covariance function decays slowly according to a power law.

To keep the notation simple and the paper easy to read, the analysis carried out in this article was restricted to spherical grains. However, we believe that this assumption can be easily relaxed to some extent following standard techniques of stochastic geometry. Another interesting open problem is to investigate how thinnings affect covariance decay properties in the light-tailed setting where the grain size distribution in the proposed Boolean model is assumed to decay rapidly.

A Regular variation

A measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is called *regularly varying* with exponent $\gamma \in \mathbb{R}$ if it is positive for all large enough input values and for all $a > 0$,

$$\frac{f(at)}{f(t)} \rightarrow a^\gamma$$

as $t \rightarrow \infty$. A regularly varying function with exponent zero is called *slowly varying*. For a good overview on the theory of regular variation, see for example [BGT87]. In this section we will summarize some key properties of regularly varying functions which are needed in the analysis. The first one is a consequence of the Karamata's theorem [BGT87].

Lemma A.1. *Let F be distribution function on \mathbb{R}_+ with a regularly varying tail of exponent $\alpha > p$. Then for any constant $a > 0$,*

$$\int_{ax}^{\infty} r^p F(dr) \sim \left(\frac{\alpha}{\alpha - p} \right) a^{-(\alpha-p)} \bar{F}(x) x^p \quad \text{as } x \rightarrow \infty.$$

Lemma A.2. *Assume that ℓ is slowly varying. Then for any $z_0 \in \mathbb{R}^d$,*

$$\ell(|z - z_0|) \sim \ell(|z|) \quad \text{as } |z| \rightarrow \infty.$$

Proof. Fix $z_0 \in \mathbb{R}^d$, and write $|z - z_0| = a_z |z|$, where $a_z = |z - z_0|/|z|$. Because $a_z \rightarrow 1$ as $|z| \rightarrow \infty$, we can fix m such that $a_z \in [1/2, 3/2]$ for $|z| > m$. Now, for any $z \in \mathbb{R}^d$ such that $|z| > m$, it follows that

$$\left| \frac{\ell(|z - z_0|)}{\ell(|z|)} - 1 \right| = \left| \frac{\ell(a_z |z|)}{\ell(|z|)} - 1 \right| \leq \sup_{a \in [1/2, 3/2]} \left| \frac{\ell(a |z|)}{\ell(|z|)} - 1 \right|.$$

The right side above tends to zero as $|z| \rightarrow \infty$ because $\ell(a|z|)/\ell(|z|) \rightarrow 1$ locally uniformly with respect to a [BGT87, Thm. 1.2.1]. \square

Lemma A.3. *Assume that f is regularly varying with exponent $-\gamma < 0$. Then there exists a constant $u > 0$ such that $f(y) \leq 2f(x)$ for all $y \geq x \geq u$.*

Proof. By the uniform convergence of regularly varying functions [BGT87, Thm. 1.5.2], $f(\lambda x)/f(x) \rightarrow \lambda^{-\gamma}$ uniformly for $\lambda \geq 1$ as $x \rightarrow \infty$. This implies that we can find $u > 0$ such that $f(\lambda x) \leq 2f(x)$ for all $x \geq u$ and all $\lambda \geq 1$. Now because $y \geq x$, we have

$$f(y) \leq \sup_{\lambda \geq 1} f(\lambda x) \leq 2f(x).$$

\square

Lemma A.4. *Let F a probability measure on \mathbb{R}_+ , and let f and g be bounded positive functions on \mathbb{R}_+ such that $f(r) \sim g(r)$ as $r \rightarrow \infty$. Then,*

$$\int_r^\infty f(s)F(ds) \sim \int_r^\infty g(s)F(ds) \quad \text{as } r \rightarrow \infty.$$

Proof.

$$\left| 1 - \frac{\int_r^\infty f(s)F(ds)}{\int_r^\infty g(s)F(ds)} \right| = \left| \frac{\int_r^\infty \left(1 - \frac{f(s)}{g(s)}\right) g(s)F(ds)}{\int_r^\infty g(s)F(ds)} \right| \leq \sup_{r \leq s} \left| 1 - \frac{f(s)}{g(s)} \right|$$

\square

B Intersections of distant balls

Lemma B.1. *Let F be a probability distribution on \mathbb{R}_+ which follows a power law with tail exponent $\alpha > d$. Then for any $r_1, r_2 \geq 0$,*

$$\int_{\mathbb{R}_+} |B_{r_1+r}(o) \cap B_{r_2+r}(z)| F(dr) \sim c_{\alpha,d} \bar{F}(|z|) |z|^d \quad \text{as } |z| \rightarrow \infty, \quad (\text{B.1})$$

where

$$c_{\alpha,d} = \int_0^\infty |B_r(o) \cap B_r(e_1)| \alpha r^{-\alpha-1} dr, \quad (\text{B.2})$$

and e_1 is the first unit vector in the standard basis of \mathbb{R}^d .

Proof. Because the Lebesgue measure is rotation-invariant, we may assume without loss of generality that $z = ue_1$ for $u > 0$. Fix $r_1, r_2 \geq 0$, and denote left side of (B.1) by $I(u)$. We will prove the claim by first showing that

$$I_1(u) = \int_{\mathbb{R}_+} |B_r(o) \cap B_r(ue_1)| F(dr) \sim cu^d \bar{F}(u), \quad (\text{B.3})$$

and then showing that the remainder $I_2(u) = I(u) - I_1(u)$ tends to zero faster than $u^d \bar{F}(u)$ as $u \rightarrow \infty$.

To prove (B.3), let F_u be the distribution of a random variable obtained by dividing a F -distributed random variable by u , so that $F_u(r) = F(ur)$. Then a change of variables shows that

$$I_1(u) = u^d \int_{\mathbb{R}_+} |B_{r/u}(o) \cap B_{r/u}(e_1)| F(dr) = u^d \int_{\mathbb{R}_+} \phi(r) F_u(dr),$$

where $\phi(r) = |B_r(0) \cap B_r(e_1)|$. Because ϕ is continuous, $r^{-d} \phi(r) \leq |B_1|$ for all $r > 0$, and $\phi(r) = 0$ for $r \leq 1/2$, we may apply [KLNS07, Lemma 2] (with $p = d$, $\gamma = \alpha$, $q = \alpha + 1$) to obtain

$$\int_{\mathbb{R}_+} \phi(r) F_u(dr) \sim \bar{F}(u) \int_0^\infty \phi(r) \alpha r^{-\alpha-1} dr = c \bar{F}(u),$$

which implies the validity of (B.3).

To show that $I_2(u)$ tends to zero faster than $u^d \bar{F}(u)$, note first that for all $u > 2(r_1 + r_2)$,

$$I_2(u) = \int_{u/4}^\infty r^d \psi_u(r) F(dr) \leq \left(\sup_{r > u/4} \psi_u(r) \right) \int_{u/4}^\infty r^d F(dr), \quad (\text{B.4})$$

where

$$\psi_u(r) = |B_{r_1/r+1}(o) \cap B_{r_2/r+1}(e_1 u/r)| - |B_1(o) \cap B_1(e_1 u/r)|.$$

The equality in (B.4) follows because $\psi_u(r) = 0$ when $u > 2(r_1 + r_2)$ and $r < u/4$.

Note that by Lemma A.1, the integral on the right side of (B.4) is asymptotically equivalent to

$$\int_{u/4}^\infty r^d F(dr) \sim \left(\frac{\alpha}{\alpha - d} \right) 4^{\alpha-d} \bar{F}(u) u^d.$$

In light of (B.4), it hence suffices to show that

$$\sup_{r > u/4} \psi_u(r) \rightarrow 0 \quad (\text{B.5})$$

as $u \rightarrow \infty$. This will be done by inspecting the geometry of ψ_u . Because the intersection of the unit-balls above is a subset of the intersection of the larger balls, we can bound the nonnegative function ψ_u using the annuli around the unit-balls, so that

$$\begin{aligned} \psi_u(r) &= |(B_{r_1/r+1}(o) \cap B_{r_2/r+1}(e_1 u/r)) \setminus (B_1(o) \cap B_1(e_1 u/r))| \\ &\leq |B_{r_1/r+1}(o) \setminus B_1(o)| + |B_{r_2/r+1}(e_1 u/r) \setminus B_1(e_1 u/r)| \\ &= |B_1|((r_1/r + 1)^d - 1) + |B_1|((r_2/r + 1)^d - 1) \\ &\leq 2|B_1| \left(\left(\frac{r_1 \vee r_2}{r} + 1 \right)^d - 1 \right). \end{aligned}$$

Because this bound is valid for all u , we conclude (B.5), and the proof is complete. \square

Lemma B.2. *Let F be a probability distribution on \mathbb{R}_+ which follows a power law with tail exponent $\alpha > d$. Then there exist constants $u > 0$ and $c > 0$ such that*

$$\int_{\mathbb{R}_+} |B_{r_1+r}(o) \cap B_{r_2+r}(z)| F(dr) \leq c \bar{F}(|z|) |z|^d \quad (\text{B.6})$$

whenever $|z| > u$ and $r_1 + r_2 \leq |z|/2$.

Proof. Observe first using Lemma A.1 that

$$\int_{|z|/4}^{\infty} r^d F(dr) \sim c_1 |z|^d \bar{F}(|z|),$$

where $c_1 = 4^{\alpha-d} \alpha / (\alpha - d)$. Hence, we may fix a constant $u > 0$ such that

$$\int_{|z|/4}^{\infty} r^d F(dr) \leq 2c_1 |z|^d \bar{F}(|z|) \quad (\text{B.7})$$

whenever $|z| > u$.

Assume now that $|z| > u$ and $r_1 + r_2 \leq |z|/2$. In this case the intersection on the left side of (B.6) is nonempty only when $r > |z|/4$. For any such $r > |z|/4$, a crude estimate shows that

$$|B_{r_1+r}(o) \cap B_{r_2+r}(z)| \leq |B_{r_1+r}(o)| \leq |B_1|(r_1 + r_2 + r)^d,$$

which together with the inequality $r_1 + r_2 \leq |z|/2 < 2r$ shows that

$$|B_{r_1+r}(o) \cap B_{r_2+r}(z)| \leq 3^d |B_1| r^d.$$

As a consequence,

$$\int_{\mathbb{R}_+} |B_{r_1+r}(o) \cap B_{r_2+r}(z)| F(dr) \leq 3^d |B_1| \int_{|z|/4}^{\infty} r^d F(dr),$$

so that by virtue of (B.7), the claim holds for $c = 2c_1 3^d |B_1|$. \square

Lemma B.3. *Let F be a probability distribution on \mathbb{R}_+ which follows a power law with tail exponent $\alpha > d$, and let h be an arbitrary positive function. Then there exist constants $u > 0$ and $c > 0$ such that*

$$(|z|^d \bar{F}(|z|))^{-1} \int_{\mathbb{R}_+} |B_r(o) \cap B_r(z)| h(r) F(dr) \leq c \sup_{r \geq |z|/2} h(r)$$

for $|z| > u$.

Proof. By using Lemma B.2, fixing the constants u and c as in the lemma, and noting that the integrand vanishes for $r \leq |z|/2$, we see that

$$\int_{\mathbb{R}_+} |B_r(o) \cap B_r(z)| h(r) F(dr) \leq h^*(z) \int_{\mathbb{R}_+} |B_r(o) \cap B_r(z)| F(dr) \leq c h^*(z) \bar{F}(|z|) |z|^d$$

for all $|z| > u$, where $h^*(z) = \sup_{r \geq |z|/2} h(r)$. \square

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